

# Existence of an endogenously complete equilibrium driven by a diffusion

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## Abstract

The existence of complete Radner equilibria is established in an economy which parameters are driven by a diffusion process. Our results complement those in the literature. In particular, we work under essentially minimal regularity conditions and treat time-inhomogeneous case.

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## 1 Introduction

The three basic topics in asset pricing theory are arbitrage, single-agent optimality, and equilibrium; see, e.g., Duffie [6], Dana and Jeanblanc [4], and Karatzas and Shreve [9]. While in the first two cases the general mathematical theory is essentially complete, see, e.g., Delbaen and Schachermayer [5] (arbitrage) and Kramkov and Schachermayer [12] and Mostovyi [17] (optimal investment), the situation with equilibrium is more involved.

In the continuous time setting, when stocks are defined by their dividends, the first sufficient conditions for the existence of dynamic Radner

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equilibria are obtained only recently in Anderson and Raimondo [1]. A key idea is to look for a *complete* Radner equilibrium, which is then necessarily constructed in two steps. First, one gets a static Arrow-Debreu equilibrium; this is standard and requires very few assumptions, see, e.g., Dana [3] and Kramkov [13]. Then one proves the *endogenous completeness* property, that is, shows that the stock prices generated from the dividends by this Arrow-Debreu equilibrium define a complete financial market.

The novelty and difficulty are in the second step. From a pure mathematical point of view, the verification of endogenous completeness can be stated as a *backward martingale representation problem*; see Kramkov and Predoiu [14]. There are random variables  $\zeta > 0$  and  $\psi = (\psi^j)$ , which are explicitly constructed in terms of model's primitives: dividends, incomes, and utilities. A probability measure  $\mathbb{Q}$  and a  $\mathbb{Q}$ -martingale  $S = (S_t^j)$  are defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const } \zeta \quad \text{and} \quad S_t^j = \mathbb{E}_{\mathbb{Q}}[\psi^j | \mathcal{F}_t].$$

The problem is to verify whether every  $\mathbb{Q}$ -martingale can be written as a stochastic integral with respect to  $S$ .

The results of Anderson and Raimondo [1] have been generalized in Hugonnier et al. [7] and Riedel and Herzberg [18]. We build on these works. Similar to these papers we assume that the parameters of the economy have a Markov-type dependence on a diffusion process, that the Jacobian matrix of terminal dividends is non-degenerated, and that the dependence on time is analytic. Our contribution is two-fold.

First, we work under essentially minimal regularity conditions. For instance, with respect to the state variable, the volatility matrix of underlying diffusion is only uniformly continuous, while the drift vector is just measurable. A detailed comparison between our assumptions and those in the literature is given in Remarks 3.2, 3.3, 3.4, and 3.7.

Second, we rigorously treat the case of time-dependence in dividends, income streams, and utilities. For instance, we allow the utility functions for intermediate consumption to be of the form:  $e^{-\nu^m t} u_m(c)$  with *different* impatience coefficients  $\nu^m$ . Although the papers [1] and [18] claim to cover such time-inhomogeneous situations, their arguments are, at best, incomplete; see Remarks 3.4 and 3.7.

## Basic concepts and notations

If  $x$  and  $y$  are vectors in  $\mathbb{R}^n$ , then  $xy$  denotes the scalar product and  $|x| \triangleq \sqrt{xx}$ . If  $a \in \mathbb{R}^{m \times n}$  is a matrix with  $m$  rows and  $n$  columns, then  $ax$  denotes

its product on the (column-)vector  $x$ ,  $a^*$  stands for the transpose, and  $|a| \triangleq \sqrt{\text{trace}(aa^*)}$ .

Let  $\mathbf{X}$  be a Banach space and  $D$  be a set in an Euclidean space  $\mathbb{R}^d$  contained in the closure of its interior. A map  $f : D \rightarrow \mathbf{X}$  is *analytic* if for every  $x \in D$  there exist a number  $\epsilon(x) > 0$  and elements  $(A_\alpha(x))$  in  $\mathbf{X}$  such that

$$f(y) = \sum_{\alpha} A_\alpha(x)(y - x)^\alpha, \quad y \in D, |y - x| < \epsilon(x),$$

where the series converges in the norm  $\|\cdot\|_{\mathbf{X}}$  of  $\mathbf{X}$ , the summation is taken with respect to multi-indices  $\alpha = (\alpha_1, \dots, \alpha_d)$  of non-negative integers, and, for  $x = (x_1, \dots, x_d)$ ,  $x^\alpha \triangleq \prod_{i=1}^d x_i^{\alpha_i}$ . A map  $f : D \rightarrow \mathbf{X}$  is *Hölder continuous* if there is  $0 < \delta < 1$  such that

$$\sup_{x, y \in D, x \neq y} \frac{\|f(x) - f(y)\|_{\mathbf{X}}}{|x - y|^\delta} < \infty.$$

Of course, a map  $f : D \rightarrow \mathbf{X}$  is *bounded* if  $\sup_{x \in D} \|f(x)\|_{\mathbf{X}} < \infty$ .

In this paper,  $\mathbf{X}$  is one of the following spaces of functions defined on a set  $E \subset \mathbb{R}^d$ :

$\mathbf{L}_\infty = \mathbf{L}_\infty(E) = \mathbf{L}_\infty(E, dx)$ : the Lebesgue space of bounded real-valued functions  $f$  on  $E$  with the norm  $\|f\|_{\mathbf{L}_\infty} \triangleq \text{ess sup}_{x \in E} |f(x)|$ .

$\mathbf{C} = \mathbf{C}(E)$ : the Banach space of bounded and continuous real-valued functions  $f$  on  $E$  with the norm  $\|f\|_{\mathbf{C}} \triangleq \sup_{x \in E} |f(x)|$ .

## 2 Complete Radner equilibrium

We consider an economy with  $M$  agents. The uncertainty and the information flow are modeled by a complete filtered probability space  $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ . The agents choose cumulative consumption processes  $C = (C_t)_{t \in [0,1]}$  from a set  $\mathcal{C}$  of optional non-decreasing processes. Their preferences regarding consumption are specified in terms of (expected) utility functionals:

$$\mathbb{U}^m : \mathcal{C} \rightarrow \mathbb{R} \cup \{-\infty\} \cup \{\infty\}, \quad m = 1, \dots, M.$$

The agents receive cumulative income processes  $I^m \in \mathcal{C}$ ,  $m = 1, \dots, M$ .

The financial market consists of a zero-coupon bond and  $J$  stocks. The bond pays the notional  $\Psi > 0$  at maturity  $t = 1$ . The stocks pay the

dividend rates  $\theta = (\theta_t^j)$  and the terminal dividends  $\Theta = (\Theta^j)$ . Thus, the total cash dividend paid by  $j$ th stock up to time  $t \in [0, 1]$  is

$$D_t^j = \int_0^t \theta_u^j du + \Theta^j 1_{\{t=1\}}.$$

The prices of the assets are determined *endogenously* by the equilibrium mechanism specified in Definition 2.1.

By a  $(B, S)$ -market we call an optional process  $B = (B_t) > 0$  and a  $J$ -dimensional semimartingale  $S = (S_t^j)$  having the terminal values

$$B_1 = \Psi \quad \text{and} \quad S_1^j = \frac{\Theta^j}{B_1} + \int_0^1 \frac{\theta_u^j}{B_u} du, \quad j = 1, \dots, J. \quad (1)$$

Here  $B$  denotes the price process of the zero-coupon bond and  $S^j B$  represents the wealth process of the buy-and-hold strategy for  $j$ th stock; equivalently,  $S^j$  is the *discounted* value of this strategy.

A probability measure  $\mathbb{Q}$  is a *martingale measure for  $S$*  if  $\mathbb{Q} \sim \mathbb{P}$  and  $S$  is a  $\mathbb{Q}$ -martingale. We call a  $(B, S)$ -market *complete* if there is only one such  $\mathbb{Q}$ ; this is the case if and only if every local martingale under  $\mathbb{Q}$  is a stochastic integral with respect to  $S$ , see Jacod [8, Section XI.1(a)].

We now introduce the main object of our study.

**Definition 2.1.** A pair  $((B, S), (\hat{C}^m)_{m=1, \dots, M})$ , consisting of a  $(B, S)$ -market and consumptions  $\hat{C}^m \in \mathcal{C}$ ,  $m = 1, \dots, M$ , is a *complete Radner equilibrium* if

1. The  $(B, S)$ -market is complete; denote by  $\mathbb{Q}$  the unique martingale measure for  $S$ .
2. The *clearing condition* holds:

$$\sum_{m=1}^M \hat{C}_t^m = \sum_{m=1}^M I_t^m, \quad t \in [0, 1]. \quad (2)$$

3. For every  $m = 1, \dots, M$  we have

$$|\mathbb{U}^m(\hat{C}^m)| + \mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{dI_t^m}{B_t}\right] < \infty,$$

the consumption  $\hat{C}^m$  satisfies the *budget constraint*:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{d\hat{C}_t^m}{B_t}\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{dI_t^m}{B_t}\right],$$

and  $\mathbb{U}^m(\hat{C}^m) \geq \mathbb{U}^m(C)$  for every consumption  $C \in \mathcal{C}$  satisfying same budget constraint:

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{dC_t}{B_t}\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{dI_t^m}{B_t}\right].$$

For the convenience of future references we also recall the definition of the (static) Arrow-Debreu equilibrium.

**Definition 2.2.** A pair  $(P, (\hat{C}^m)_{m=1,\dots,M})$ , consisting of a positive optional consumption price process  $P > 0$  and consumptions  $\hat{C}^m \in \mathcal{C}$ ,  $m = 1, \dots, M$ , is an *Arrow-Debreu equilibrium* if the clearing condition (2) holds and for every  $m = 1, \dots, M$  we have

$$|\mathbb{U}^m(\hat{C}^m)| + \mathbb{E}\left[\int_0^1 P_t dI_t^m\right] < \infty,$$

the consumption  $\hat{C}^m$  satisfies the *budget constraint*:

$$\mathbb{E}\left[\int_0^1 P_t d\hat{C}_t^m\right] = \mathbb{E}\left[\int_0^1 P_t dI_t^m\right],$$

and  $\mathbb{U}^m(\hat{C}^m) \geq \mathbb{U}^m(C)$  for every consumption  $C \in \mathcal{C}$  satisfying same budget constraint:

$$\mathbb{E}\left[\int_0^1 P_t dC_t\right] = \mathbb{E}\left[\int_0^1 P_t dI_t^m\right].$$

The relations between these two types of equilibria are summarized in the following lemmas.

**Lemma 2.3.** Let  $((B, S), (\hat{C}^m)_{m=1,\dots,M})$  be a complete Radner equilibrium. Let  $\mathbb{Q}$  be the martingale measure for  $S$ , denote by  $Z$  the density process of  $\mathbb{Q}$  under  $\mathbb{P}$ , and set  $P \triangleq Z/B$ . Then  $\mathbb{E}[P_1 \Psi] = 1$  and the pair  $(P, (\hat{C}^m)_{m=1,\dots,M})$  is an Arrow-Debreu equilibrium.

**Lemma 2.4.** Let  $(P, (\hat{C}^m)_{m=1,\dots,M})$  be an Arrow-Debreu equilibrium such that  $\mathbb{E}[P_1 \Psi] = 1$ . Let  $\mathbb{Q}$  be the probability measure with the density process  $Z_t \triangleq \mathbb{E}[P_1 \Psi | \mathcal{F}_t]$  under  $\mathbb{P}$ , set  $B \triangleq Z/P$ , and suppose that the  $J$ -dimensional martingale  $S = (S_t^j)$  under  $\mathbb{Q}$  with the terminal value (1) is well-defined. If the  $(B, S)$ -market is complete, then  $((B, S), (\hat{C}^m)_{m=1,\dots,M})$  is a complete Radner equilibrium.

The proofs are direct consequences of the identities

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^1 \frac{dC_t}{B_t}\right] = \mathbb{E}[Z_1 \int_0^1 \frac{dC_t}{B_t}] = \mathbb{E}\left[\int_0^1 \frac{Z_t}{B_t} dC_t\right],$$

which hold for an optional process  $B > 0$ , an optional non-decreasing process  $C$ , and a probability measure  $\mathbb{Q} \sim \mathbb{P}$  with the density process  $Z$ .

### 3 Model's primitives

The primitives of our model are defined in terms of the  $d$ -dimensional diffusion

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad t \in [0, 1]. \quad (3)$$

Here  $X_0 \in \mathbb{R}^d$ ,  $W$  is a Brownian motion with values in  $\mathbb{R}^d$  defined on a complete filtered probability space  $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$ , and the drift  $b = b(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and the volatility  $\sigma = \sigma(t, x) : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are such that for all  $i, j = 1, \dots, d$ :

(A1) the maps  $t \mapsto b^i(t, \cdot)$  of  $[0, 1]$  to  $\mathbf{L}_\infty$  and  $t \mapsto \sigma^{ij}(t, \cdot)$  of  $[0, 1]$  to  $\mathbf{C}$  are analytic on  $(0, 1)$  and Hölder continuous on  $[0, 1]$ . For  $t \in [0, 1]$  and  $x \in \mathbb{R}^d$  the matrix  $\sigma(t, x)$  has the inverse  $\sigma^{-1}(t, x)$  and there exists a constant  $N > 0$ , same for all  $t$  and  $x$ , such that

$$|\sigma^{-1}(t, x)| \leq N. \quad (4)$$

Moreover, there exists a strictly increasing function  $\omega = (\omega(\epsilon))_{\epsilon > 0}$  such that  $\omega(\epsilon) \rightarrow 0$  as  $\epsilon \downarrow 0$  and, for all  $t \in [0, 1]$  and all  $x, y \in \mathbb{R}^d$ ,

$$|\sigma(t, x) - \sigma(t, y)| \leq \omega(|x - y|).$$

In view of (4), we can assume that the filtration  $\mathbf{F}$  is generated by  $X$ :

$$\mathbf{F} = \mathbf{F}^X \triangleq (\mathcal{F}_t^X)_{t \in [0,1]}, \quad (5)$$

where, as usual,  $\mathcal{F}_t^X$  denotes the  $\sigma$ -field generated by  $(X_s)_{s \leq t}$  and complemented with  $\mathbb{P}$ -null sets.

*Remark 3.1.* With respect to  $x$ , the conditions in (A1) are, essentially, the minimal classical assumptions guaranteeing the existence and the uniqueness of the weak solution to (3); see Stroock and Varadhan [19, Theorem 7.2.1] and Krylov [16, 15]. This weak solution is also well-defined when  $b$  and  $\sigma$  are only measurable functions with respect to  $t$ . Example 2.5 in [14] shows that the requirement on  $\sigma = \sigma(t, x)$  to be  $t$ -analytic is, however, essential for our main Theorem 4.1 to hold.

*Remark 3.2.* Let us compare our assumptions on the diffusion  $X$  with those in literature. In the pioneering paper [1],  $X$  is a Brownian motion. In [7] the conditions are imposed on the diffusion coefficients  $b = b(t, x)$  and  $\sigma = \sigma(t, x)$  and on the transition density  $p = p(t, x, s, y)$ . In the main body of [7], it is assumed that  $b$ ,  $\sigma$ , and  $p$  are analytic functions with respect to all their arguments. In the technical appendix to [7], these functions are required to be analytic with respect to  $t$  and  $s$  and 5-times (7-times for  $p$ ) continuously differentiable with respect to  $x$  and  $y$ . In [18] the diffusion coefficients  $b$  and  $\sigma$  do not depend on  $t$ , the matrix  $\sigma$  is invertible and  $b = b(x)$ ,  $\sigma = \sigma(x)$ , and  $\sigma^{-1} = \sigma^{-1}(x)$  and bounded and analytic functions.

From the point of view of applications, the most severe constraint of our setup is the boundedness assumption on the diffusion coefficients. This condition was used in the “backward” martingale representation theorem in [14], on which this paper relies, to facilitate references to the results from elliptic PDEs.

### 3.1 Notional and dividends

From now on, the uncertainty and the information flow are modeled by the filtered probability space  $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$  with the filtration  $\mathbf{F}$  generated by the diffusion  $X$  from (3).

We assume that the notional  $\Psi$ , for the bond, and the intermediate dividend rates  $\theta = (\theta_t^j)$  and the terminal dividends  $\Theta = (\Theta^j)$ , for the stocks, have the form:

$$\begin{aligned}\Psi &= G(X_1)e^{\int_0^1 q(s, X_s)ds}, \\ \theta_t^j &= f^j(t, X_t)e^{\int_0^t p^j(s, X_s)ds}, \quad t \in [0, 1], \\ \Theta^j &= G(X_1)F^j(X_1)e^{\int_0^1 p^j(s, X_s)ds}, \quad j = 1, \dots, J,\end{aligned}$$

where the functions  $G, F^j$  on  $\mathbb{R}^d$  and  $q, f^j, p^j$  on  $[0, 1] \times \mathbb{R}^d$  satisfy

(A2) The functions  $G$  and  $F^j$  are weakly differentiable,  $G > 0$ , the Jacobian matrix  $\left(F_{x_i}^j\right)_{j=1, \dots, J, i=1, \dots, d}$  has rank  $d$  almost surely under the Lebesgue measure on  $\mathbb{R}^d$ , and there is a constant  $N > 0$  such that

$$|G_{x_i}(x)| + |F_{x_i}^j(x)| \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d,$$

and such that  $t \mapsto e^{-N|x|}f^j(t, \cdot) \triangleq (e^{-N|x|}f^j(t, x))_{x \in \mathbb{R}^d}$  and  $t \mapsto q(t, \cdot)$ ,  $t \mapsto p^j(t, \cdot)$  are Hölder continuous maps of  $[0, 1]$  to  $\mathbf{L}_\infty$  whose restrictions on  $(0, 1)$  are analytic.

The expressions for  $\Psi$ ,  $\theta^j$ , and  $\Theta^j$  are similar to those in [1], where the rate functions  $q$  and  $p^j$  equal to zero.

*Remark 3.3.* When the diffusion coefficients  $\sigma^{ij}$  and  $b^i$  and the functions  $f^j$ ,  $q$ , and  $p^j$  are also  $x$ -analytic it is enough to assume that the Jacobian matrix  $(F_{x_i}^j)$  has rank  $d$  only on an open set (equivalently, at just one point), see [1], [7], and [18]. Without  $x$ -analyticity this is not possible, see Example 2.7 in [14].

*Remark 3.4.* The  $t$ -analyticity condition on  $f^j$  cannot be omitted; see Example 2.6 in [14]. We stress that  $t \mapsto e^{-N|\cdot|} f^j(t, \cdot)$  is analytic as a map of  $(0, 1)$  to  $\mathbf{L}_\infty$ ; same is true for the rate functions  $p^j$  and  $q$  and the diffusion coefficients  $b^i$  and  $\sigma^{ij}$ . This is more than just boundedness and the analyticity of  $t \mapsto e^{-N|x|} f^j(t, x)$  for every  $x \in \mathbb{R}^d$ . For instance, the map  $t \mapsto (e^{-N|x|} \sin(te^{x^2}))_{x \in \mathbb{R}^d}$  of  $(0, 1)$  to  $\mathbf{L}_\infty$  is not even differentiable.

Observe that the analyticity of the map  $t \mapsto f(t, \cdot)$  of  $[0, 1]$  to  $\mathbf{L}_\infty$  implies the existence of a bounded continuously differentiable function  $h = h(t, x)$  on  $[0, 1] \times \mathbb{R}^d$  such that  $t \mapsto h(t, \cdot)$  and  $t \mapsto h_{x_i}(t, \cdot)$  are analytic maps of  $[0, 1]$  to  $\mathbf{L}_\infty$  and

$$h(t, X_t) = \mathbb{E}\left[\int_t^1 f(s, X_s) ds | \mathcal{F}_t\right], \quad t \in [0, 1]. \quad (6)$$

This follows from the results of Komatsu [11] and Kato and Tanabe [10] on time-analyticity of solutions of evolution equations; the details of the argument can be found in [14, Sections 3 and 4]. In particular,  $h(\cdot, x)$  and  $h_{x_i}(\cdot, x)$  are analytic functions on  $[0, 1]$  for every  $x \in \mathbb{R}^d$ ; this property plays a crucial role in the proof of our main result.

Let me point out that in [1] and [18] and also in the technical appendix to [7] it is assumed instead that  $f = f(t, x)$  is a real analytic function on  $[0, 1] \times \mathbf{R}^d$  in the usual pointwise sense; for simplicity of argument, suppose also that  $f$  is a bounded function. The proofs in these papers then rely on the claim that the bounded function  $h = h(t, x)$  satisfying (6) is a real analytic function on  $[0, 1] \times \mathbf{R}^d$ . However, the arguments behind this claim are either missing or incomplete. It is not clear to me whether such assertion holds even if  $X$  is a standard Brownian motion.

### 3.2 Preferences and endowments

The agents consume continuously on  $[0, 1]$  according to an optional process  $\xi = (\xi_t) \geq 0$  of consumption rates and also at maturity  $t = 1$  according

to a random variable  $\Xi \geq 0$  of terminal wealth. The process of cumulative consumption is thus given by

$$C_t = \int_0^t \xi_s ds + \Xi 1_{\{t=1\}}, \quad t \in [0, 1].$$

The expected utility of  $m$ th agent has the form:

$$\mathbb{U}^m(C) \triangleq \mathbb{E}\left[\int_0^1 u^m(t, \xi_t, X_t) e^{-\int_0^t r(s, X_s) ds} dt + U^m(\Xi, X_1) e^{-\int_0^1 r(t, X_t) dt}\right],$$

where  $r = r(t, x)$  is the “impatience” rate, common among the agents, and  $u^m = u^m(t, c, x)$  and  $U^m = U^m(c, x)$  are utility functions for intermediate and terminal consumptions defined for  $t \in [0, 1]$ ,  $c \geq 0$ , and  $x \in \mathbb{R}^d$ . These expressions are similar to those in [1], where the impatience rate  $r = r(t, x)$  does not depend on  $x$ .

The income process of  $m$ th agent is given by

$$I_t^m = \int_0^t \lambda_s^m ds + \Lambda^m 1_{\{t=1\}}, \quad t \in [0, 1],$$

where the optional process  $\lambda^m$  of income rates and the random variable  $\Lambda^m$  of terminal endowment satisfy

$$\lambda^m \geq 0, \quad \Lambda^m \geq 0, \quad \text{and } \mathbb{P}[I_1^m > 0] > 0, \quad m = 1, \dots, M. \quad (7)$$

The total terminal and intermediate incomes are denoted by

$$\Lambda \triangleq \sum_{m=1}^M \Lambda^m \quad \text{and} \quad \lambda \triangleq \sum_{m=1}^M \lambda^m.$$

We shall say that a function  $f = f(c)$  on  $[0, \infty)$  satisfies *the Inada conditions* if  $f$  is strictly concave, strictly increasing, and continuously differentiable on  $(0, \infty)$  and  $\lim_{c \downarrow 0} f_c(c) = \infty$ ,  $\lim_{c \rightarrow \infty} f_c(c) = 0$ . Moreover,  $f(0) = \lim_{c \downarrow 0} f(0)$ ; this limit may equal  $-\infty$ .

We impose the following conditions on  $r$ ,  $(U^m)$ , and  $\Lambda$ :

(A3)  $t \mapsto r(t, \cdot)$  is a Hölder continuous map of  $[0, 1]$  to  $\mathbf{L}_\infty$  whose restriction on  $(0, 1)$  is analytic.

(A4) For  $x \in \mathbb{R}^d$ , the terminal wealth utility function  $U^m(\cdot, x)$  on  $[0, \infty)$  satisfies the Inada conditions. On  $(0, \infty) \times \mathbb{R}^d$  the derivatives  $U_{cc}^m$  and  $U_{cx}^m$  exist and are continuous,  $U_{cc}^m < 0$ , and, for some constant  $N > 0$ ,

$$|U^m(1, x)| \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d, \quad (8)$$

$$\left(-\frac{cU_{cc}^m}{U_c^m} + \frac{|U_{cx}^m|}{U_c^m}\right)(c, x) \leq N, \quad (c, x) \in (0, \infty) \times \mathbb{R}^d. \quad (9)$$

(A5)  $\Lambda = e^{H(X_1)}$ , where the function  $H = H(x)$  is weakly differentiable, and, for some constant  $N > 0$ ,

$$|H(x)| \leq N(1 + |x|) \quad \text{and} \quad |H_{x^i}| \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d.$$

*Remark 3.5.* In the state-homogeneous case, where  $U^m(c, x) = U^m(c)$ , the inequality (8) holds trivially, while (9) means the boundedness of the risk-aversion coefficient  $-cU_{cc}^m/U_c^m$ . Theorem 5.1 below shows that the families of functions  $U = U(c, x)$  satisfying (A4) are convex cones closed under superconvolution with respect to  $c$ .

The assumptions on the utility functions  $u^m = u^m(t, c, x)$  for intermediate consumption are bundled with the conditions on the total income rate  $\lambda$ . We assume that

$$\lambda_t = e^{h(t, X_t)}, \quad t \in [0, 1],$$

and that the functions  $u^m$  and  $h$  are either time homogeneous:

(A6)  $u^m(t, c, x) = u^m(c, x)$ . For  $x \in \mathbb{R}^d$  the function  $u^m(\cdot, x)$  on  $[0, \infty)$  satisfies the Inada conditions. There is a constant  $N > 0$  such that

$$|u^m(e^y, x)| \leq e^{N(1+|x|+|y|)}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}. \quad (10)$$

(A7)  $h(t, x) = h(x)$  and has a linear growth: for some  $N \geq 0$ ,

$$|h(x)| \leq N(1 + |x|), \quad x \in \mathbb{R}^d.$$

or they satisfy

(A8) For  $(t, x) \in [0, 1] \times \mathbb{R}^d$  the function  $u^m(t, \cdot, x)$  satisfies the Inada conditions. The derivatives  $u_{ct}^m$  and  $u_{cc}^m$  exist and  $u_{cc}^m < 0$ . There are a constant  $N > 0$  such that

$$|u^m(t, e^y, x)| \leq e^{N(1+|x|+|y|)}, \quad (t, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}, \quad (11)$$

and an open set  $V \subset (0, \infty)^2$  containing  $(0, 1) \times \{1\}$ , that is a neighborhood of  $(0, 1) \times \{1\}$ , such that

$$(t, s) \mapsto (g(t, se^y, x))_{(x, y) \in \mathbb{R}^d \times \mathbb{R}},$$

is a bounded analytic map of  $V$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ , where  $g = g(t, c, x)$  stands for  $u_{ct}^m/u_c^m$ ,  $cu_{cc}^m/u_c^m$ , and  $u_c^m/(cu_{cc}^m)$ .

(A9)  $h(t, x) = h_1(t, x) + h_2(x)$ , where  $t \mapsto h_1(t, \cdot)$  is a Hölder continuous map of  $[0, 1]$  to  $\mathbf{L}_\infty(\mathbb{R}^d)$  whose restriction on  $(0, 1)$  is analytic and the function  $h_2 = h_2(x)$  has a linear growth: for some  $N \geq 0$ ,

$$|h_2(x)| \leq N(1 + |x|), \quad x \in \mathbb{R}^d.$$

The role of either pair of these assumptions is to imply the assertions of Lemma 6.3 for every aggregate utility function  $u(w) = u(t, c, x; w)$  introduced in Section 4.

*Remark 3.6.* A classical example of a utility function in (A8) is

$$u^m(t, c, x) \triangleq e^{\nu^m(t)} \frac{c^{1-a^m} - 1}{1 - a^m} g^m(x), \quad (12)$$

where  $a^m$  is a positive constant of risk-aversion,  $\nu^m = \nu^m(t)$  is an analytic function on  $(0, 1)$  with bounded derivative, and  $g^m = g^m(x)$  is a positive function with exponential growth:  $0 < g^m(x) \leq N e^{N|x|}$ , for some  $N > 0$ ; if  $a^m = 1$ , then, by continuity,  $u^m(t, c, x) \triangleq e^{\nu^m(t)} (\ln c) g^m(x)$ .

Theorem 5.1 shows that the families of functions  $u = u(t, c, x)$  satisfying either (A6) or (A8) are convex cones closed under sup-convolution with respect to  $c$ . In particular, one can begin with functions as in (12) and build more general utility functions by successively taking positive linear combinations and applying sup-convolutions with respect to  $c$ .

*Remark 3.7.* Regarding (A8) we observe that for a function  $g = g(t, c, x)$  the analyticity of the map  $(t, s) \mapsto (g(t, se^y, x))_{(x,y) \in \mathbb{R}^d \times \mathbb{R}}$  of a neighborhood of  $(0, 1) \times \{1\}$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$  readily implies the analyticity of the map  $(t, c) \rightarrow g(t, c, \cdot)$  of  $(0, 1) \times (0, \infty)$  to  $\mathbf{L}_\infty(\mathbb{R}^d)$ . The inverse is not true. For example, the function

$$g(c) \triangleq \sin(\ln^2(c)), \quad c > 0,$$

is analytic and uniformly bounded on  $(0, \infty)$ . However, the map  $s \mapsto (g(se^y))_{y \in \mathbb{R}}$  taking values in  $\mathbf{L}_\infty(\mathbb{R}^d)$  is not even continuous at  $s = 1$ :

$$\begin{aligned} \limsup_{s \rightarrow 1} \sup_{y \in \mathbb{R}} |g(se^y) - g(e^y)| &= \limsup_{s \rightarrow 1} \sup_{y \in \mathbb{R}} |\sin(\ln^2(se^y)) - \sin(\ln^2(e^y))| \\ &= \limsup_{\epsilon \rightarrow 0} \sup_{y \in \mathbb{R}} |\sin((\epsilon + y)^2) - \sin(y^2)| = 2. \end{aligned}$$

Let me point out that in [1] and [18], instead of (A8) and (A9), it is assumed that the functions  $u^m = u^m(t, c, x)$  and  $h = h(t, x)$  are just pointwise real analytic. However, the proofs in these papers rely on the ambiguous argument discussed at the end of Remark 3.4. Hence, it is not clear to me whether such pointwise analyticity is sufficient.

## 4 Main result

Denote by  $\Sigma^M$  the simplex in  $\mathbb{R}^M$ :

$$\Sigma^M \triangleq \{w \in [0, \infty)^M : \sum_{m=1}^M w^m = 1\}.$$

For a weight  $w \in \Sigma^M$  define the aggregate utility functions  $U(w) = U(c, x; w)$  and  $u(w) = u(t, c, x; w)$ , where  $(t, c, x) \in [0, 1] \times (0, \infty) \times \mathbb{R}^d$ , as the  $w$ -weighted sup-convolutions with respect to  $c$ :

$$U(c, x; w) \triangleq \sup \left\{ \sum_{m=1}^M w^m U^m(c^m, x) : c^m \geq 0, c^1 + \cdots + c^M = c \right\},$$

$$u(t, c, x; w) \triangleq \sup \left\{ \sum_{m=1}^M w^m u^m(t, c^m, x) : c^m \geq 0, c^1 + \cdots + c^M = c \right\}.$$

Theorem 5.1 shows that the aggregate utility functions  $U(w)$  and  $u(w)$  satisfy same conditions (A4), (A6), and (A8) as  $U^m$  and  $u^m$ ; in particular, they satisfy the Inada conditions with respect to  $c$ .

By  $(\Pi^m(w))_{m=1,\dots,M}$  we denote the  $w$ -weighted Pareto allocation of  $\Lambda$ :

$$\Pi^m(w) \triangleq 0 \quad \text{if } w^m = 0,$$

$$w^m U_c^m(\Pi^m(w), X_1) \triangleq U_c(\Lambda, X_1; w) \quad \text{if } w^m > 0,$$

and by  $(\pi^m(w))_{m=1,\dots,M}$  the optional processes of Pareto consumption rates:

$$\pi^m(w) \triangleq 0 \quad \text{if } w^m = 0,$$

$$w^m u_c^m(t, \pi_t^m(w), X_t) \triangleq u_c(t, \lambda_t, X_t; w), \quad t \in [0, 1], \quad \text{if } w^m > 0.$$

The cumulative Pareto consumption processes are given by

$$C_t^m(w) \triangleq \int_0^t \pi_s^m(w) ds + \Pi^m(w) 1_{\{t=1\}}, \quad t \in [0, 1], \quad m = 1, \dots, M. \quad (13)$$

We denote by  $\mathcal{W}$  the subset of  $\Sigma^M$  such that

$$\mathcal{W} \triangleq \{w \in \Sigma^M : \Phi^m(w) = 0, m = 1, \dots, M\},$$

where the function  $\Phi^m$  on  $\Sigma^M$  is given by

$$\Phi^m(w) \triangleq \mathbb{E} \left[ e^{-\int_0^1 r(t, X_t) dt} U_c(\Lambda, X_1; w) (\Pi^m(w) - \Lambda^m) \right. \\ \left. + \int_0^1 e^{-\int_0^t r(s, X_s) ds} u_c(t, \lambda_t, X_t; w) (\pi_t^m(w) - \lambda_t^m) dt \right].$$

These functions are well-defined; see Lemma 6.4.

For  $w \in \Sigma^M$  define the martingale  $Y(w)$  by

$$Y_t(w) \triangleq \mathbb{E}[\Psi U_c(\Lambda, X_1; w) e^{-\int_0^t r(s, X_s) ds} | \mathcal{F}_t], \quad t \in [0, 1],$$

and the probability measure  $\mathbb{Q}(w) \sim \mathbb{P}$  by

$$\frac{d\mathbb{Q}(w)}{d\mathbb{P}} \triangleq \frac{Y_1(w)}{Y_0(w)}.$$

We also denote by  $B(w)$  the positive optional process

$$B_t(w) \triangleq \frac{Y_t(w)}{u_c(t, \lambda_t, X_t; w)} e^{\int_0^t r(s, X_s) ds} 1_{\{t < 1\}} + \Psi 1_{\{t=1\}}, \quad t \in [0, 1],$$

and by  $S(w) = (S_t^j(w))$  the  $J$ -dimensional  $\mathbb{Q}(w)$ -martingale with the terminal value

$$S_1^j(w) \triangleq \frac{\Theta^j}{\Psi} + \int_0^1 \frac{\theta_u^j}{B_u(w)} du, \quad j = 1, \dots, J.$$

The integrability conditions needed for the existence of such  $Y(w)$  and  $S(w)$  are verified in Lemmas 6.5 and 6.6.

We now state the main result of the paper.

**Theorem 4.1.** *Let the conditions (5), (7), (A1)–(A5), and either (A6) and (A7) or (A8) and (A9) hold. Then there is a complete Radner equilibrium. The set  $\mathcal{W}$  is not empty and belongs to the interior of  $\Sigma^M$ . Every complete Radner equilibrium has the form  $((B(w), S(w)), C(w))$  for some  $w \in \mathcal{W}$  and, conversely,  $((B(w), S(w)), C(w))$  is a complete Radner equilibrium for every  $w \in \mathcal{W}$ .*

The proof is given in Section 6 and relies on the criteria for the existence of Arrow-Debreu equilibria from [13], on the “backward” martingale representation result from [14], and on the study of utility functions in Section 5 below.

## 5 Convex cones of utility functions closed under sup-convolution

Let  $\mathcal{U}$  be a family of real-valued (utility) functions  $u = u(t, c, x)$  on  $[0, 1] \times (0, \infty) \times \mathbb{R}^d$  which are concave with respect to  $c$ . We are interested in  $\mathcal{U}$  being a convex cone closed under the operations of sup-convolution with

respect to  $c$ : for every  $u$ ,  $u_1$ , and  $u_2$  in  $\mathcal{U}$  and every constant  $a > 0$  the functions  $au$ ,  $u_1 + u_2$ ,  $(u_1 \oplus_c u_2)$  belong to  $\mathcal{U}$ , where

$$(u_1 \oplus_c u_2)(t, c, x) \triangleq \sup\{u_1(t, c_1, x) + u_2(t, c_2, x) : c_1 > 0, c_1 + c_2 = c\}. \quad (14)$$

Motivated by the proof of Theorem 4.1 this property will be established for the following families of functions:

$\mathcal{U}_1$  consists of measurable functions  $u = u(t, c, x)$  on  $[0, 1] \times (0, \infty) \times \mathbb{R}^d$  which satisfy the Inada conditions with respect to  $c$  and such that, for some constant  $N = N(u) > 0$ ,

$$|u(t, e^y, x)| \leq e^{N(1+|x|+|y|)}, \quad (t, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}. \quad (15)$$

$\mathcal{U}_2$  consists of functions  $u \in \mathcal{U}_1$  such that the derivatives  $u_{cc}$  and  $u_{cx^i}$  exist and are continuous functions with respect to  $(c, x)$ ,  $u_{cc} < 0$ , and, for some constant  $N = N(u) > 0$ ,

$$\left(-\frac{cu_{cc}}{u_c} + \frac{|u_{cx^i}|}{u_c}\right)(t, c, x) \leq N, \quad (t, c, x) \in [0, 1] \times (0, \infty) \times \mathbb{R}^d. \quad (16)$$

$\mathcal{U}_3$  consists of functions  $u \in \mathcal{U}_1$  such that the derivatives  $u_{ct}$  and  $u_{cc}$  exist,  $u_{cc} < 0$ , and

$$(t, s) \mapsto (g(t, se^y, x))_{(x,y) \in \mathbb{R}^d \times \mathbb{R}} \quad (17)$$

is a bounded analytic map of some neighborhood of  $(0, 1) \times \{1\}$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ , where  $g = g(t, c, x)$  stands for  $u_{ct}/u_c$ ,  $cu_{cc}/u_c$ , and  $u_c/(cu_{cc})$ .

**Theorem 5.1.** *Each of the families  $\mathcal{U}_i$ ,  $i = 1, 2, 3$ , is a convex cone closed under sup-convolution with respect to  $c$ .*

The rest of this section is devoted to the proof of this theorem which we divide into lemmas. In the study of  $\mathcal{U}_3$  we use the versions of composition and implicit function theorems for analytic maps with values in  $\mathbf{L}_\infty$  stated in Appendix A.

**Lemma 5.2.** *Each of the families  $\mathcal{U}_i$ ,  $i = 1, 2, 3$ , is a convex cone.*

*Proof.* For  $\mathcal{U}_1$  and  $\mathcal{U}_2$  the result is straightforward.

Let  $u^1$  and  $u^2$  be in  $\mathcal{U}_3$ , denote  $u \triangleq u^1 + u^2$ , and fix  $t_0 \in (0, 1)$ . The analyticity of the map (17) at the point  $(t_0, 1)$ , with  $g = g(t, c, x)$  standing for  $u_{ct}/u_c$  and  $cu_{cc}/u_c$ , readily follows if we can show that

$$(t, s) \mapsto \left(w(t, se^y, x) \triangleq \frac{u_c^1}{u_c^1 + u_c^2}(t, se^y, x)\right)_{(x,y) \in \mathbb{R}^d \times \mathbb{R}} \quad (18)$$

is an analytic map of some neighborhood of  $(t_0, 1)$  to  $\mathbf{L}_\infty$ .

For  $i = 1, 2$  denote  $a_i \triangleq -cu_{cc}^i/u_c^i$  and  $q_i \triangleq u_{ct}^i/u_c^i$ . Observe that

$$f_i(t, sc, x) \triangleq \frac{u_c^i(t, sc, x)}{u_c^i(t_0, c, x)} = \exp\left(-\int_1^s \frac{1}{r} a_i(t, rc, x) dr + \int_{t_0}^t q_i(r, c, x) dr\right),$$

$$w(t, sc, x) = \frac{w(t_0, c, x) f_1(t, sc, x)}{w(t_0, c, x) f_1(t, sc, x) + (1 - w(t_0, c, x)) f_2(t, sc, x)}.$$

The analyticity of  $a_i$  and  $q_i$  in  $\mathcal{U}_3$  and Theorem A.1 yield the analyticity of the map  $(t, s) \mapsto (f_i(t, se^y, x))_{(x,y) \in \mathbb{R}^{d+1}}$  of some neighborhood of  $(t_0, 1)$  to  $\mathbf{L}_\infty$ . As  $f_i(t_0, c, x) = 1$  and

$$(p, q) \mapsto \left( \frac{w(t_0, e^y, x)p}{w(t_0, e^y, x)p + (1 - w(t_0, e^y, x))q} \right)_{(x,y) \in \mathbb{R}^d \times \mathbb{R}}$$

is an analytic map of  $(0, \infty)^2$  to  $\mathbf{L}_\infty$ , Theorem A.1 yields the required analyticity of the map (18); hence, also the analyticity of the map (17) at  $(t_0, 1)$  with  $g = g(t, c, x)$  standing for  $u_{ct}/u_c$  and  $cu_{cc}/u_c$ .

As  $a \triangleq -cu_{cc}/u_c$  is a convex combination of the corresponding risk-aversions for  $u^1$  and  $u^2$ , there is  $N > 0$  such that  $\frac{1}{N} \leq a \leq N$ . Another application of Theorem A.1 implies the analyticity of the map (17) at  $(t_0, 1)$  with  $g \triangleq 1/a$ .  $\square$

**Lemma 5.3.** *Let  $u^1, u^2 \in \mathcal{U}_1$ . Then  $u \triangleq u^1 \oplus_c u^2$  belongs to  $\mathcal{U}_1$  and the upper bound in (14) is attained on  $\hat{c}_1 = f$  and  $\hat{c}_2 = c - f$ , where the function  $f = f(t, c, x)$  is such that  $0 < f < c$  and*

$$u_c(t, c, x) = u_c^1(t, f(t, c, x), x) = u_c^2(t, c - f(t, c, x), x). \quad (19)$$

*Proof.* Elementary arguments show that  $u(t, \cdot, x)$  satisfies the Inada conditions and that the upper bound in (14) is attained on the above  $\hat{c}_1$  and  $\hat{c}_2$ . Since

$$(u^1 + u^2)(t, c/2, x) \leq u(t, c, x) \leq (u^1 + u^2)(t, c, x)$$

we deduce that  $u$  satisfies (15). Hence,  $u$  belongs to  $\mathcal{U}_1$ .  $\square$

**Lemma 5.4.** *Let  $u^1, u^2 \in \mathcal{U}_2$ . Then  $u \triangleq u^1 \oplus_c u^2$  belongs to  $\mathcal{U}_2$  and*

$$\frac{u_c}{u_{cc}}(t, c, x) = \frac{u_c^1}{u_{cc}^1}(t, f, x) + \frac{u_c^2}{u_{cc}^2}(t, c - f, x), \quad (20)$$

$$\frac{u_{cx^i}}{u_{cc}}(t, c, x) = \frac{u_{cx^i}^1}{u_{cc}^1}(t, f, x) + \frac{u_{cx^i}^2}{u_{cc}^2}(t, c - f, x), \quad (21)$$

where  $f = f(t, c, x)$  is given by (19).

*Proof.* By Lemma 5.3,  $u \in \mathcal{U}_1$ . From (19) and the implicit function theorem we deduce that the derivatives  $f_c$  and  $f_{x^i}$  and, then,  $u_{cc}$  and  $u_{cx^i}$  exist and are continuous functions with respect to  $(c, x)$ . Direct computations show that  $u_{cc} < 0$  and that the identities (20) and (21) hold. It follows that  $u_c/(cu_{cc})$  and  $u_{cx^i}/u_c$  are convex combinations of the corresponding coefficients of  $u^1$  and  $u^2$ . This implies (16). Hence,  $u \in \mathcal{U}_2$ .  $\square$

The verification of the closure of  $\mathcal{U}_3$  under  $\oplus_c$  relies on

**Lemma 5.5.** *Let  $u^1, u^2 \in \mathcal{U}_3$ ,  $u \triangleq u^1 \oplus_c u^2$ , and the function  $f = f(t, c, x)$  on  $[0, 1] \times (0, \infty) \times \mathbb{R}^d$  be given by (19). Let  $t_0 \in (0, 1)$ . Then*

$$(t, s) \mapsto \left( \frac{f(t, se^y, x)}{f(t_0, e^y, x)}, \frac{se^y - f(t, se^y, x)}{e^y - f(t_0, e^y, x)} \right)_{(x,y) \in \mathbb{R}^d \times \mathbb{R}}$$

are analytic maps of a neighborhood of the point  $(t_0, 1)$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ .

*Proof.* For  $t \in [0, 1]$ ,  $s > 0$ ,  $r > 0$ ,  $y \in \mathbb{R}$ , and  $x \in \mathbb{R}^d$  denote

$$\begin{aligned} h_1(t, s, y, x) &\triangleq \frac{f(t, se^y, x)}{se^y - f(t, se^y, x)}, \\ h_2(t, s, y, x) &\triangleq \frac{h_1(t, s, y, x)}{h_1(t_0, 1, y, x)}, \\ h_3(r, y, x) &\triangleq \frac{1 + h_1(t_0, 1, y, x)}{1 + rh_1(t_0, 1, y, x)} \end{aligned}$$

and observe that  $h_i > 0$  and

$$\begin{aligned} f(t, se^y, x) &= se^y \frac{h_1}{1 + h_1} = sf(t_0, e^y, x)h_2h_3(h_2, y, x), \\ se^y - f(t, se^y, x) &= se^y \frac{1}{1 + h_1} = s(e^y - f(t_0, e^y, x))h_3(h_2, y, x), \end{aligned} \tag{22}$$

where we omitted the argument  $(t, s, y, x)$  for  $h_1$  and  $h_2$ . As  $h_2(t_0, 1, \cdot, \cdot) = 1$  and  $r \mapsto h_3(r, \cdot, \cdot)$  is an analytic map of  $(0, \infty)$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ , the result follows from (22) and Theorem A.1 in the appendix if

$$(t, s) \mapsto (h_2(t, s, y, x))_{(x,y) \in \mathbb{R}^d \times \mathbb{R}} \tag{23}$$

is an analytic map of a neighborhood of the point  $(t_0, 1)$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ .

We rely on the version of implicit function theorem stated in Theorem A.2. Define the function

$$h_4(t, s, r, y, x) \triangleq \ln \left( \frac{u_c^1(t, sf(t_0, e^y, x)rh_3, x)}{u_c^2(t, s(e^y - f(t_0, e^y, x))h_3, x)} \right), \tag{24}$$

where  $h_3 = h_3(r, y, x)$ . From (22) and (19) we deduce that

$$h_4(t, s, h_2, y, x) = \ln \left( \frac{u_c^1(t, f(t, se^y, x), x)}{u_c^2(t, se^y - f(t, se^y, x), x)} \right) = 0.$$

Denote  $a_i \triangleq -cu_{cc}^i/u^c$  and  $q_i \triangleq u_{ct}^i/u_c^i$ . Direct computations show that

$$\begin{aligned} \frac{\partial h_4}{\partial t} &= q_1 - q_2, \\ \frac{\partial h_4}{\partial s} &= \frac{1}{s}(-a_1 + a_2), \\ \frac{\partial h_4}{\partial r} &= -a_1 \frac{\partial}{\partial r} \ln(rh_3) + a_2 \frac{\partial}{\partial r} \ln h_3 \\ &= -\frac{1}{r} \left( a_1 \frac{1}{1+rh_1} + a_2 \frac{rh_1}{1+rh_1} \right), \end{aligned}$$

where  $h_1$  is evaluated at  $(t_0, 1, y, x)$  and the omitted arguments for  $a_i$  and  $q_i$  are as for  $u_c^i$  in (24). Another application of Theorem A.1 yields that each of these partial derivatives for  $h_4$  defines an analytic map of some neighborhood of  $(t_0, 1, 1)$  to  $\mathbf{L}_\infty$ . As  $h_4(t_0, 1, 1, y, x) = 0$  we obtain that  $(t, s, r) \mapsto h_4(t, s, r, \cdot, \cdot)$  is also an analytic map of a neighborhood of  $(t_0, 1, 1)$  to  $\mathbf{L}_\infty$ . Moreover, there is  $N > 0$  such that

$$\frac{1}{N} \leq -\frac{\partial h_4}{\partial y}(t_0, 1, 1, \cdot, \cdot) \leq N.$$

The required analyticity of the map (23) follows now from Theorem A.2.  $\square$

The following lemma completes the proof of the theorem.

**Lemma 5.6.** *Let  $u^1, u^2 \in \mathcal{U}_3$ . Then  $u \triangleq u^1 \oplus_c u^2$  belongs to  $\mathcal{U}_3$ .*

*Proof.* Let  $f = f(t, c, x)$  be defined by (19). From (19) and the implicit function theorem we obtain that the derivatives  $f_t$  and  $u_{ct}$  exist and

$$\frac{u_{ct}}{u_{cc}}(t, c, x) = \frac{u_{ct}^1}{u_{cc}^1}(t, f, x) + \frac{u_{ct}^2}{u_{cc}^2}(t, c - f, x). \quad (25)$$

Denote  $a \triangleq -cu_{cc}/u_c$  and  $q \triangleq u_{ct}/u_c$  and let  $a_i$  and  $q_i$  be the corresponding coefficients for  $u^i$ ,  $i = 1, 2$ . From (20) and (25) we deduce that

$$\begin{aligned} \frac{1}{a} &= \frac{1}{a_1(t, f, x)} \frac{f}{c} + \frac{1}{a_2(t, c - f, x)} \frac{c - f}{c}, \\ \frac{q}{a} &= \frac{q_1}{a_1}(t, f, x) \frac{f}{c} + \frac{q_2}{a_2}(t, c - f, x) \frac{c - f}{c}. \end{aligned}$$

The required boundedness and analyticity of the maps

$$(t, s) \mapsto (g(t, se^y, x))_{(x,y) \in \mathbb{R}^d \times \mathbb{R}}, \quad g \text{ stands for } a, 1/a, \text{ and } q,$$

of some neighborhood of  $(0, 1) \times \{1\}$  to  $\mathbf{L}_\infty$  follows now from the boundedness and analyticity of these maps for  $u^1$  and  $u^2$  and from Lemma 5.5 and Theorem A.1.  $\square$

## 6 Proof of Theorem 4.1

As in [1], [7], and [18] the proof consists of two steps. First, in Lemma 6.4, we describe all Arrow-Debreu equilibria. Then, in Lemma 6.6, we obtain that every Arrow-Debreu equilibrium yields a complete Radner equilibrium.

We always work under the assumptions of Theorem 4.1. We shall often use the fact that, as the diffusion coefficients  $b = b(t, x)$  and  $\sigma = \sigma(t, x)$  are bounded, the running maximum  $\sup_{t \in [0,1]} |X_t|$  has all exponential moments.

We begin with some estimates concerning the utility functions.

**Lemma 6.1.** *Let  $U = U(c, x)$  be a function on  $(0, \infty) \times \mathbb{R}^d$  satisfying (A4). Then there is a constant  $N > 0$  such that, for  $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ ,*

$$(|U| + U_c - U_{cc} + |U_{cx^i}|)(e^y, x) \leq e^{N(1+|x|+|y|)}. \quad (26)$$

*Proof.* From (A4) we deduce that the function  $G(x, y) \triangleq \ln U_c(e^y, x)$  has bounded derivatives. This yields the estimate for  $U_c$ . The inequalities for  $U_{cc}$  and  $U_{cx^i}$  then follow from (9), while the inequality for  $U$  follows from (8).  $\square$

**Lemma 6.2.** *Let  $u = u(c, x)$  be a function on  $(0, \infty) \times \mathbb{R}^d$  satisfying (A6). Then there is a constant  $N > 0$  such that*

$$u_c(e^y, x) \leq e^{N(1+|x|+|y|)}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}.$$

*Proof.* From the concavity of  $u(\cdot, x)$  we deduce that

$$u_c(c, x) \leq \frac{2}{c}\{u(c, x) - u(c/2, x)\} \leq \frac{2}{c}\{|u(c, x)| + |u(c/2, x)|\}$$

and the result follows from (10).  $\square$

**Lemma 6.3.** *Let  $u = u(t, c, x)$  be a function on  $[0, 1] \times (0, \infty) \times \mathbb{R}^d$  satisfying (A8) and such that for every  $(c, x) \in (0, \infty) \times \mathbb{R}^d$  the function  $u_c(\cdot, c, x)$  on*

$[0, 1]$  is continuous. Let  $h = h(t, x)$  be a function on  $[0, 1] \times \mathbb{R}^d$  satisfying (A9). Then there is a constant  $N > 0$  such that

$$t \mapsto e^{-N|\cdot|} u_c(t, e^{h(t,\cdot)}, \cdot) \triangleq (e^{-N|x|} u_c(t, e^{h(t,x)}, x))_{x \in \mathbb{R}^d} \quad (27)$$

is a Hölder continuous map of  $[0, 1]$  to  $\mathbf{L}_\infty$  whose restriction on  $(0, 1)$  is analytic.

*Proof.* From the concavity of  $u(t, \cdot, x)$  we obtain

$$u_c(t, c, x) \leq \frac{2}{c}(u(t, c, x) - u(t, c/2, x))$$

and then (11) implies the existence of  $N > 0$  such that

$$u_c(t, e^y, x) \leq e^{N(1+|x|+|y|)}, \quad (t, x, y) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}. \quad (28)$$

Fix  $t_0 \in (0, 1)$ . Denote  $a \triangleq -cu_{cc}/u_c$  and  $q \triangleq u_{ct}/u_c$  and observe that

$$\frac{u_c(t, sc, x)}{u_c(t_0, c, x)} = \exp \left( - \int_1^s \frac{1}{r} a(t, rc, x) dr + \int_{t_0}^t q(r, c, x) dr \right).$$

The analyticity of  $a$  and  $q$  in (A8), Theorem A.1, and the inequality (28) yield the existence of  $N > 0$  such that

$$(t, s) \mapsto \left( e^{-N(|x|+|y|)} u_c(t, se^y, x) \right)_{(x,y) \in \mathbb{R}^d \times \mathbb{R}} \quad (29)$$

is an analytic map of a neighborhood of  $(0, 1) \times \{1\}$  to  $\mathbf{L}_\infty(\mathbb{R}^{d+1})$ . Define the functions

$$\begin{aligned} g_1(t, s, x) &\triangleq u_c(t, se^{h(t_0,x)}, x), \\ g_2(t, x) &\triangleq e^{h(t,x)-h(t_0,x)} = e^{h_1(t,x)-h_1(t_0,x)}. \end{aligned}$$

From the analyticity of (29) and the linear growth for  $h(t_0, \cdot)$  we deduce the existence of  $N > 0$  such that  $(t, s) \mapsto e^{-N|\cdot|} g_1(t, s, \cdot)$  is an analytic map of a neighborhood of  $(t_0, 1)$  to  $\mathbf{L}_\infty$ . We also obtain that  $g_2 > 0$ ,  $g_2(t_0, \cdot) = 1$ , and, by (A9) and Theorem A.1, the map  $t \mapsto g_2(t, \cdot)$  of  $(0, 1)$  to  $\mathbf{L}_\infty$  is analytic. As

$$u_c(t, e^{h(t,x)}, x) = g_1(t, g_2(t, x), x),$$

Theorem A.1 now implies the analyticity of the map (27) at  $t_0$ ; hence, also the analyticity of this map on  $(0, 1)$ .

It remains to verify the Hölder continuity of the map (27). Observe first that as the function  $u_c(\cdot, c, x)$  on  $[0, 1]$  is continuous and the function  $u_c(t, \cdot, x)$  on  $(0, \infty)$  is continuous and decreasing, the map  $t \mapsto u_c(t, \cdot, x)$ ,  $t \in [0, 1]$ , is uniformly continuous on compact sets: for every  $n > 0$

$$\sup_{|y| \leq n} |u_c(t, e^y, x) - u_c(s, e^y, x)| \rightarrow 0 \quad \text{if } |s - t| \rightarrow 0, s, t \in [0, 1].$$

This property and the continuity of the function  $h(\cdot, x)$  on  $[0, 1]$  implies the continuity of the function  $u_c(\cdot, e^{h(\cdot, x)}, x)$  on  $[0, 1]$  for every  $x \in \mathbb{R}^d$ .

Fix  $0 < s < t < 1$  and denote

$$f(r) = u_c(s + r(t - s), e^{h(s, x) + r(h(t, x) - h(s, x))}, x), \quad r \in [0, 1].$$

From (28), (A8), and (A9) we deduce the existence of constants  $M, N > 0$  and  $0 < \delta < 1$  such that, for  $r \in (0, 1)$ ,

$$\begin{aligned} f(r) &\leq e^{N(1+|x|)}, \\ |f'(r)| &\leq M((t-s) + |h(t, x) - h(s, x)|) f(r) \leq e^{N(1+|x|)} |t-s|^\delta. \end{aligned}$$

It follows that

$$|u_c(t, e^{h(t, x)}, x) - u_c(s, e^{h(s, x)}, x)| = |f(1) - f(0)| \leq e^{N(1+|x|)} |t-s|^\delta.$$

This implies the Hölder continuity of the map (27) taking values in  $\mathbf{L}_\infty$  on  $(0, 1)$ . As, for every  $x \in \mathbb{R}^d$ , the function  $u_c(\cdot, e^{h(\cdot, x)}, x)$  on  $[0, 1]$  is continuous, we also obtain the Hölder continuity of this map on  $[0, 1]$ .  $\square$

The following lemma accomplishes the first step. For  $w \in \Sigma^M$  denote

$$\eta(w) \triangleq e^{-\int_0^1 r(t, X_t) dt} U_c(\Lambda, X_1; w) \Lambda + \int_0^1 e^{-\int_0^t r(s, X_s) ds} u_c(t, \lambda_t, X_t; w) \lambda_t dt.$$

**Lemma 6.4.** *An Arrow-Debreu equilibrium exists. We have*

$$\mathbb{E} \left[ \sup_{w \in \Sigma^M} \eta(w) \right] < \infty$$

and, hence, the functions  $(\Phi^m)$  are well-defined. The set  $\mathcal{W}$  is not empty and belongs to the interior of  $\Sigma^M$ . The set of Arrow-Debreu equilibria is given by  $(yP(w), (C^m(w))_{m=1, \dots, M})$ ,  $y > 0$ ,  $w \in \mathcal{W}$ , where the consumptions  $C^m(w)$  are defined in (13) and the consumption price process  $P(w)$  is given by

$$\begin{aligned} P_t(w) &\triangleq e^{-\int_0^t r(s, X_s) ds} u_c(t, \lambda_t, X_t; w) 1_{\{t < 1\}} \\ &\quad + e^{-\int_0^1 r(s, X_s) ds} U_c(\Lambda, X_1; w) 1_{\{t=1\}}, \quad t \in [0, 1]. \end{aligned} \tag{30}$$

*Proof.* We use the criteria for the existence of Arrow-Debreu equilibria from [13]. Theorem 1.4, Corollary 1.5, Theorem 2.4, and Lemma 3.1 in this paper imply the assertions of the lemma if, for  $z \in (0, 1]$  and  $m = 1, \dots, M$ ,

$$\mathbb{E} \left[ \int_0^1 e^{-\int_0^t r(s, X_s) ds} |u^m(t, z\lambda_t, X_t)| dt + e^{-\int_0^1 r(s, X_s) ds} |U^m(z\Lambda, X_1)| \right] < \infty.$$

From the estimates (10) and (11) for  $u^m$  and (26) for  $U^m$  we deduce the existence of a constant  $N > 0$  such that for every  $0 < z \leq 1$

$$\begin{aligned} |u^m(t, z\lambda_t, X_t)| + |U^m(z\Lambda, X_1)| &= |u^m(t, ze^{h(t, X_t)}, X_t)| + |U^m(ze^{H(X_1)}, X_1)| \\ &\leq \frac{1}{z^N} e^{N(1+\sup_{t \in [0,1]} |X_t|)}. \end{aligned}$$

The result now follows from the boundedness of the impatience rate function  $r = r(t, x)$  and from the existence of all exponential moments for  $\sup_{t \in [0,1]} |X_t|$ .  $\square$

Hereafter, for  $w \in \Sigma^M$ , we denote by  $P(w)$  the consumption price process defined in (30).

**Lemma 6.5.** *For every  $w \in \Sigma^M$  we have*

$$\mathbb{E}[P_1(w)\Psi] = \mathbb{E}[e^{-\int_0^1 r(t, X_t) dt} U_c(\Lambda, X_1; w)\Psi] < \infty;$$

in particular, the martingale  $Y(w)$  is well-defined.

*Proof.* From Lemma 6.1 we deduce the existence of  $N > 0$  such that

$$|U^m(e^y, x)| \leq e^{N(1+|x|+|y|)}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R},$$

and, then, from Theorem 5.1 that  $U(w) = U(c, x; w)$  satisfies (A4). Lemma 6.1, the linear growth of  $H = H(x)$ , and the boundedness of  $r = r(t, x)$  imply the existence of  $N > 0$  such that

$$|P_1(w)| = U_c(e^{H(X_1)}, X_1) e^{-\int_0^1 r(t, X_t) dt} \leq e^{N(1+|X_1|)}.$$

From (A2) we deduce a similar estimate for  $\Psi$ :

$$\Psi = G(X_1) e^{\int_0^1 q(s, X_s) ds} \leq e^{N(1+|X_1|)}.$$

The integrability of  $P_1(w)\Psi$  follows now from the existence of all exponential moments for  $|X_1|$ .  $\square$

Lemmas 2.3, 6.4, and 6.5 imply that the set of complete Radner equilibria is a subset of  $((B(w), S(w)), C(w))$ ,  $w \in \mathcal{W}$ . In view of Lemma 2.4, it only remains to be shown that for every  $w \in \mathcal{W}$  the  $(B(w), S(w))$ -market is complete. This is accomplished in

**Lemma 6.6.** *For every  $w \in \Sigma^M$  the  $\mathbb{Q}(w)$ -martingale  $S(w) = (S_t^j(w))$  is well-defined and  $\mathbb{Q}(w)$  is its only martingale measure.*

*Proof.* We rely on the martingale representation result from [14]. Recall that

$$\begin{aligned} \frac{d\mathbb{Q}(w)}{d\mathbb{P}} &= \frac{Y_1(w)}{Y_0(w)}, \\ Y_1(w) &= K(X_1; w) e^{\int_0^1 \beta(t, X_t) dt}, \\ S_1^j(w) &= F^j(X_1) e^{\int_0^1 \alpha^j(t, X_t) dt} + \int_0^1 \frac{g^j(t, X_t; w)}{Y_t(w)} e^{\int_0^t (\alpha^j + \beta)(s, X_s) ds} dt, \end{aligned}$$

where

$$\begin{aligned} \beta(t, x) &= q(t, x) - r(t, x), \\ \alpha^j(t, x) &= p^j(t, x) - q(t, x), \\ K(x; w) &= G(x) U_c(e^{H(x)}, x; w), \\ g^j(t, x; w) &= f^j(t, x) u_c(t, e^{h(t, x)}, x; w). \end{aligned}$$

According to Theorem 2.3 in [14] the result follows if the diffusion coefficients  $b$  and  $\sigma$  satisfy (A1) and there is a constant  $N = N(w) > 0$  such that

- (i) The functions  $F^j$  and  $K$  are weakly differentiable,  $K$  is strictly positive, the Jacobian matrix  $\left(F_{x^i}^j\right)_{i=1,\dots,d, j=1,\dots,J}$  has rank  $d$  almost surely under the Lebesgue measure on  $\mathbb{R}^d$ , and

$$|F_{x^i}^j(x)| + |K_{x^i}(x; w)| \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d. \quad (31)$$

- (ii) The maps  $t \mapsto e^{-N|\cdot|} g^j(t, \cdot; w) \triangleq (e^{-N|x|} g^j(t, x; w))_{x \in \mathbf{R}^d}$ ,  $t \mapsto \alpha^j(t, \cdot)$ , and  $t \mapsto \beta(t, \cdot)$  of  $[0, 1]$  to  $\mathbf{L}_\infty$  are analytic on  $(0, 1)$  and Hölder continuous on  $[0, 1]$ .

The required properties of  $F^j$ ,  $\alpha^j$ , and  $\beta$  follow immediately from the conditions of the theorem.

Clearly,  $K > 0$ . By chain rule,

$$\begin{aligned} K_{x^i}(x; w) &= G_{x^i}(x)U_c(e^{H(x)}, x; w) \\ &\quad + G(x) \left( U_{cc}(e^{H(x)}, x; w)e^{H(x)}H_{x^i}(x) + U_{cx^i}(e^{H(x)}, x; w) \right). \end{aligned}$$

From (A2) we deduce the existence of a constant  $N > 0$  such that

$$|G_{x^i}(x)| + G(x) \leq e^{N(1+|x|)}, \quad x \in \mathbb{R}^d.$$

Theorem 5.1 implies that  $U(w)$  satisfies (A4). Then, by Lemma 6.1, there is  $N > 0$  such that

$$(|U_c| + |U_{cc}| + |U_{cx^i}|)(e^y, x; w) \leq e^{N(1+|x|+|y|)}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}.$$

These inequalities and the condition (A5) for  $H$  imply the exponential estimate (31) for  $K_{x^i}$ .

Theorem 5.1 implies that  $u(w) = u(t, x, ; w)$  satisfies same conditions (A6) or (A8) as  $u^m$ . The assertion (ii) for  $g^j$  follows now from Lemmas 6.2 and 6.3 and the properties of  $f^j$  in (A2).

A careful reader may notice that in Lemma 6.3, in addition to (A8), we assumed the continuity of  $u_c(\cdot, c, x)$  on  $[0, 1]$ . This does not restrict any generality. Indeed, denoting  $q \triangleq u_{ct}/u_c$  we obtain that

$$u_c(t, c, x) = u_c(1/2, c, x)e^{\int_{1/2}^t q(r, c, x)dr}$$

and, as  $q$  is bounded,  $u_c(\cdot, c, x)$  can be continuously extended on  $[0, 1]$ .  $\square$

## A On analytic functions with values in $\mathbf{L}_\infty$

In this appendix we state versions of composition and implicit function theorems for analytic functions with values in  $\mathbf{L}_\infty$  used in the proofs of Theorems 4.1 and 5.1. Hereafter,  $\mathbf{L}_\infty \triangleq \mathbf{L}_\infty(E)$  for some fixed set  $E \subset \mathbb{R}^d$ .

**Theorem A.1.** *Let  $f = f(x, y)$  and  $g_i = g_i(x, z)$ ,  $i = 1, \dots, m$ , where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^m$ , and  $z \in \mathbb{R}^l$ , be analytic maps of neighborhoods of  $(0, 0)$  to  $\mathbf{L}_\infty$  such that  $g_i(0, 0) = 0$ . Then*

$$h(x, z) \triangleq f(x, g_1(x, z), \dots, g_m(x, z)), \quad (x, z) \in \mathbb{R}^n \times \mathbb{R}^l,$$

*is an analytic map of a neighborhood of  $(0, 0)$  to  $\mathbf{L}_\infty$ .*

**Theorem A.2.** Let  $f = f(x, y)$ , where  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}$ , be an analytic map of a neighborhood of  $(0, 0)$  to  $\mathbf{L}_\infty$  such that  $f(0, 0) = 0$  and  $f_y(0, 0) \geq \epsilon$  for some constant  $\epsilon > 0$ . Then there is a neighborhood  $V \subset \mathbb{R}^n$  of 0 and an analytic map  $g = g(x)$  of  $V$  to  $\mathbf{L}_\infty$  such that

$$f(x, g(x)) = 0, \quad x \in V.$$

The proofs of both theorems are essentially identical to the proofs of the corresponding results for real-valued analytic functions, see, e.g., Propositions IV.5.5.1 and IV.5.6.1 in [2].

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